Exact renormalization group equations from perturbative quantum gravity

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Contents

I will show that getting well-defined and non-perturbative beta functions in QG is more than just a dream.

Brief review of renormalization group

As an example, consider a fermion loop effect in QED,

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\big[\gamma^\mu(\partial_\mu - A_\mu) - im\big]\psi.$$

With the one-loop correction, we get, approximately,

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}\Big[1 - \beta \ln\Big(-\frac{\Box + m^2}{\mu^2}\Big)\Big] F^{\mu\nu}.$$

In the IR, when (Euclidean) momentum is $k^2 \ll m^2$, this becomes an irrelevant redefinition of e. However, in the UV, when $k^2 \gg m^2$, there is an effective running:

$$e^2(k) = \frac{e_0^2}{1 - \beta \ln \frac{k^2}{\mu^2}}. \qquad \beta = \frac{e^2}{6\pi^2}.$$

Thus, we avoid working with explicit non-localities of the effective action and explore interpolation between the UV and IR.

In QED, the running is an observable effect. The question is how to apply this method to quantum gravity (QG).

Gauge invariant renormalizability in QG

Covariant action of gravity: $S = \int d^4x \sqrt{-g}\, \mathcal{L}(g_{\mu\nu})\,.$

$\mathcal{L}(g_{\mu\nu})$ **can be** $\mathcal{L}(g_{\mu\nu}) = -\kappa^{-2}(R + 2\Lambda)$ **or another.**

The gauge transformation $\delta g_{\mu\nu} = R_{\mu\nu,\alpha}\xi^\alpha$**. The gauge invariance:**

$$\frac{\delta S}{\delta g_{\mu\nu}}\, R_{\mu\nu,\alpha} = 0\,.$$

Renormalizability in quantum Gravity (QG):

K. Stelle, Phys. Rev. D (1977).

General proof using Batalin-Vilkovisky formalism:

P.M. Lavrov, I.Sh., Gauge invariant renormalizability of quantum gravity, arXiv:1902.04687; PRD.

Textbook-level introduction: *I.L. Buchbinder and I.Sh., Introduction to Quantum Field Theory with Applications to Quantum Gravity (Oxford Univ. Press, 2021).*

The Faddeev-Popov approach, with $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$

$$S_{tot} = S(h) + \frac{1}{2}\chi^{\alpha} Y_{\alpha\beta}\chi^{\beta} + \frac{1}{2}\bar{C}^{\alpha} M^{\beta}_{\alpha} C_{\beta}, \quad M^{\beta}_{\alpha} = \frac{\delta\chi^{\alpha}}{\delta h_{\mu\nu}} R_{\mu\nu,\alpha}\,.$$

The useful choices of gauge fixing conditions and the weight function depend on the theory, e.g.,

Fock-deDonder gauge $\chi^{\mu} = \partial_{\nu}\Phi^{\mu\nu}, \;\; \kappa\Phi^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu},$

harmonic gauge $\chi_{\mu} = \partial^{\nu} h_{\mu\nu} - \beta\partial_{\mu} h\,, \quad \beta = \frac{1}{2},$

background gauge $\chi_{\mu} = \nabla^{\nu} h_{\mu\nu} - \beta\nabla_{\mu} h.$

Independent on the parametrization and gauge fixing, one can prove that the divergent part of effective action, $\Gamma_{div} = \Gamma_{div}(g_{\mu\nu})$, **in all loop orders, is local and satisfies the gauge identity**

$$\frac{\delta\Gamma_{div}}{\delta g_{\mu\nu}} R_{\mu\nu,\alpha} = 0\,.$$

This doesn't guarantee the multiplicative renormalizability, which requires, additionally, a "correct" power counting.

Power counting in QG

The power counting of a diagram with an arbitrary number of external lines $h_{\mu\nu}$ and number of their derivatives $d(G)$ is defined by the superficial degree of divergence $\omega(G)$,

$$\omega(G) + d(G) = \sum_{l_{int}} (4 - r_l) - 4V + 4 + \sum_V K_V .$$

The first sum is over internal lines of the diagram, r_l is the inverse power of momentum in the propagator of the given internal line, and V is the number of vertices. K_V is the number of derivatives acting on the given vertex.

In addition, there is the topological relation $L = I - V + 1$.

As the first example, consider quantum GR.

$$S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g}\,(R + 2\Lambda)\,.$$

In this case we get $\omega(G) + d(G) = 2 + 2L - 2K_\Lambda$.

Divergences in QG based on Einstein gravity

With $\Lambda = 0$, at the one-loop level there are divergences

$$\mathcal{O}(R^2_{...}) \,=\, R^2_{\mu\nu\alpha\beta}\,, \;\; R^2_{\mu\nu}\,, \;\; R^2$$

and no sign of the renormalization of the R- term or Λ.

t'Hooft and Veltman; Deser and van Nieuwenhuisen, (1974); ...

At the two-loop level we have, with $\Lambda = 0$, the divergences

$$\mathcal{O}(R^3_{...}) = R_{\mu\nu}\Box R^{\mu\nu}\,,\;...\,R^3\,,\;\; R_{\mu\nu}R^\mu_\alpha R^{\alpha\nu}\,,\;\; R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\rho\sigma}R^{\mu\nu\rho\sigma}\;.$$

M.H. Goroff and A. Sagnotti, NPB 266 (1986).

Since the last structure does not vanish on-shell, the theory is not renormalizable. A relevant observation is, however, that

for $\Lambda \neq 0$, at the two-loop level we have Einstein-Hilbert type divergences $\sim \dfrac{\Lambda}{M_P^2}$ and the Λ- type divergences $\sim \dfrac{\Lambda^2}{M_P^4}$, etc.

Within the standard perturbative approach, the lack of renormalizability means the theory has no predictive power.

Every time we introduce a new type of a counterterm, it is necessary to fix renormalization condition and this means a measurement. So, before making a single predictions, it is necessary to have an infinite amount of experimental data.

What are the possible solutions?

- **Change standard perturbative approach to something else.**
- **Change the theory, i.e., take another theory to construct QG.**

We shall explore both these approaches and show how one can explore the exact running of $1/G$ and Λ beyond the framework of the loop expansion.

First we consider the superrenormalizable versions of QG and then the effective low-energy QFT approach to the running.

The basic renormalizable option is the four-derivative model:

$$S_{QG} \;=\; -\int d^4x\sqrt{-g}\Big\{\frac{1}{\kappa^2}\,(R+2\Lambda)+\frac{1}{2\lambda}\,C^2+\frac{\omega}{3\lambda}\,R^2 \\ +\;\text{surface terms}\Big\},$$

where $$C^2(4)=R^2_{\mu\nu\alpha\beta}-2R^2_{\alpha\beta}+R^2/3\,.$$

Propagators of metric and ghosts behave like $\mathcal{O}(k^{-4})$ and we have K_4, K_2, K_0 vertices.

The superficial degree of divergence is

$$\omega(G)\,+\,d(G)\;=\;4\,-\,2K_2\,-\,4K_\Lambda.$$

The theory is renormalizable. Counterterms dimensions: $4,\ 2, 0.$

K. Stelle, Phys. Rev. D (1977).

Bad part: The tree-level spectrum includes a massive spin-2 **"ghost" with negative kinetic energy and huge mass.**

Including more than four derivatives:

$$S = \int d^4x \sqrt{-g} \Big\{ - \frac{1}{\kappa^2}\,(R + 2\Lambda) + \sum_{n=0}^{N} \Big[\omega_n^C C_{\mu\nu\alpha\beta} \Box^n C_{\mu\nu\alpha\beta} + \omega_n^R R \Box^n R \Big] \,+\, \mathcal{O}\big(R^3_{\dots}\big) \Big\}.$$

Simple analysis shows that in this theory massive ghost-like states are still present.

M. Asorey, J.-L. Lopez & I. Sh., IJMPhA (1997), hep-th/9610006.

Independent on this, the power counting is:

$$\omega(G) + d(G) \,=\, 4 \,+\, N(1 - L)\,.$$

Consider log. divergences, $\omega(G) = 0$ **with the maximal power.**

Starting from $N = 1$**, the theory is superrenormalizable, as the divergences are present only for** $L = 1, 2, 3$**.**

For $N \geq 3$ **the divergences exist only for** $L = 1$ **(one-loop).**

Exact β-functions in QG

In the superrenormalizable QG one can get the exact renormalization group equations, derived at the one-loop level !

M. Asorey, J.-L. Lopez & I. Sh., IJMPhA (1997), hep-th/9610006.

$$\beta_\Lambda = \mu \frac{d\rho_\Lambda}{d\mu} = \frac{1}{(4\pi)^2} \left(\frac{5\omega_{N-2,C}}{\omega_{N,C}} + \frac{\omega_{N-2,R}}{\omega_{N,R}} - \frac{5\omega_{N-1,C}^2}{2\omega_{N,C}^2} - \frac{\omega_{N-1,R}^2}{2\omega_{N,R}^2} \right).$$

L. Modesto, L. Rachwal & I.Sh., arXiv:1704.03988, Eur.Phys.J. C78.

$$\beta_G = \mu \frac{d}{d\mu} \left(-\frac{1}{16\pi G} \right) = -\frac{1}{6(4\pi)^2} \left(\frac{5\omega_{N-1,C}}{\omega_{N,C}} + \frac{\omega_{N-1,R}}{\omega_{N,R}} \right).$$

Different from four-derivative quantum gravity, these β-functions do not depend on the choice of gauge-fixing condition.

These beta functions do not depend on the gauge fixing or parametrization of quantum metric. And, owing to the power counting, for $N \geq 3$, these beta functions are exact.

What we know for sure

is that the higher derivatives bring heavy degrees of freedom, that can be normal fields modes, ghost modes, tachyons, and/or tachyonic ghosts.

Also, we know that at least part of the contributions of the massive degrees of freedom decouple in the IR.

The natural (and probably correct) assumption is that below the typical mass scale of the massive degrees of freedom there remain only quantum effects of the massless graviton, that means the effective QG based on Einstein's gravity.

J.F. Donoghue, Phys. Rev. D 50 (1994) 3874.

From this perspective, the exact running which we met in the superrenormalizable QG, may take place only at the energy scale above the greater mass in the spectrum of QG.

This means, typically, above the Planck scale.

Running in the effective QG

How to explore the running in the framework of effective QFT, e.g., in the effective QG?

First of all, all elements of Feynman technique, e.g., propagators and vertices, should be constructed on the basis of GR.

This does not mean there are no other terms in the action, but they play a "passive" role, like the vacuum gravitational action in the semiclassical approach.

Without the cosmological term, we have only the renormalization in higher and higher derivative sectors – boring to explore.

But, what about QG with the nonzero cosmological constant?

In this case, yes - we have all kinds of divergences and certainly can construct the renormalization group equations which make sense and are potentially interesting.

Gauge-fixing dependence

For the QG based on GR with the cosmological constant, we meet another serious problem.

The gauge and parametrization ambiguities in the running of Newton and cosmological constants are typical for the fourth-derivative QG and, to the same extent, in quantum GR.

In quantum GR, only the on-shell quantities are well-defined.

R.E. Kallosh, O.V. Tarasov, I.V. Tyutin, Nucl. Phys. B137 (1978).

This situation gives rise to the on-shell renormalization group for the dimensionless ratio of G and Λ , namely, $\lambda = 16\pi G\Lambda$:

$$\mu\frac{d\lambda}{\mu} \;=\; -\frac{29}{5}\,\frac{\lambda^2}{(4\pi)^2}.$$

E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B201 (1982) 469.

One can analyse the situation with gauge and parametrization dependencies without explicit calculations, using the general QFT theorems, see, e.g.,

I.Y. Aref'eva, A.A. Slavnov, L.D. Faddeev, Theor. Math. Phys. (1974).

B.L. Voronov, P.M. Lavrov, I.V. Tyutin, Sov. J. Nucl. Phys. (1982).

This formalism was applied to quantum gravity in

E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B201 (1982) 469.

I.Sh, A. Jacksenaev, PLB 324 (1994) 284.

Also, it was confirmed by explicit calculations, e.g., in

M. Kalmykov, Class. Quant. Grav. 12 (1995) 1401.

J. Gonçalves, T. de Paula Netto, I.Sh., PRD 97 (2018), 1712.03338.

In GR-based QG, the one-loop divergences are

$$\Gamma^{(1)}_{div} = \frac{1}{\epsilon}\int d^4x\sqrt{-g}\{c_1\, R^2_{\mu\nu\alpha\beta} + c_2 R^2_{\alpha\beta} + c_3 R^2 + c_4 \Box R + c_5 R + c_6\},$$

The gauge-fixing and parametrization dependencies vanish on shell. According to Weinberg's theorem $\Gamma^{(1)}_{div}$ is local. Thus,

$$\Gamma^{(1)}_{div}(\alpha_i) - \Gamma^{(1)}_{div}(\alpha^0_i) = \frac{1}{\epsilon}\int d^4x\sqrt{-g}\,\Big(b_1 R_{\mu\nu} + b_2 R g_{\mu\nu} + b_3 g_{\mu\nu}\Lambda + b_4 g_{\mu\nu}\Box + b_5 \nabla_\mu\nabla_\nu\Big)\,\varepsilon^{\mu\nu}\,,$$

where $b_k = b_k(\alpha_i)$ and α_i represent the full set of parameters: gauge-fixing and parametrization of quantum metric. Also,

$$\varepsilon^{\mu\nu} = \frac{1}{\sqrt{-g}}\,\frac{\delta S}{\delta g_{\mu\nu}} \sim R^{\mu\nu} - \frac{1}{2}\,(R + 2\Lambda)\,g^{\mu\nu}.$$

The two invariant quantities are

$$c_1 \quad \text{and} \quad c_{\rm inv} = c_6 - 4\Lambda c_5 + 4\Lambda^2 c_2 + 16\Lambda^2 c_3\,.$$

In particular, there is no well-defined running of the physically interesting terms, e.g., ρ_Λ, R, R^2 in this effective framework.

How can we get an invariant definition of RG in QG?

The best solution is based on the Vilkovisky–DeWitt (VdW) scheme for constructing effective action in quantum gravity.

*G.A. Vilkovisky, Nucl. Phys. B***234** *(1984) 125.*

A.O. Barvinsky and G.A. Vilkovisky, Phys. Repts. 119 (1985) 1.

B.S. DeWitt, The effective action, (Essays in honor of the sixtieth birthday of E.S. Fradkin, 1987).

We need that (at least) one-loop divergences do not depend on the gauge-fixing and parametrization of the quantum metric.

Then, we can achieve the universal running of the coefficients of $\rho_\Lambda,\ R,\ R^2$ **and, in fact, of all other terms of the action.**

And the VdW approach makes it possible, and even gives more

T. Taylor and G. Veneziano, Nucl. Phys. B ***345*** *(1990).*

B. Giacchini, T. de Paula Netto, I.Sh., JHEP (2020), 2009.04122.

Let us briefly review this interesting QFT-based construction.

Consider the parametrization dependence. The one-loop expression for the non-gauge theory with the classical action $S[\Phi_i]$ **has the form**

$$\bar{\Gamma}^{(1)} = \frac{i}{2}\,\mathrm{Ln\,Det}\,S''_{ij}, \qquad \textbf{where} \qquad S''_{ij} = \frac{\delta^2 S}{\delta\Phi_i\,\delta\Phi_j}\,.$$

Changing the variables $\Phi_i = \Phi_i(\Phi'_k)$**, we meet**

$$\bar{\Gamma}^{(1)} = \mathrm{Ln\,Det}\left(\frac{\delta^2 S}{\delta\Phi'_l\,\delta\Phi'_k}\right) = \mathrm{Ln\,Det}\left(S''_{ij}\cdot\frac{\delta\Phi_i}{\delta\Phi'_k}\,\frac{\delta\Phi_j}{\delta\Phi'_l} + \frac{\delta S}{\delta\Phi_i}\frac{\delta^2\Phi_i}{\delta\Phi'_l\,\delta\Phi'_k}\right).$$

One can see, that the two one-loop results coincide on the classical equations of motion (on shell), when

$$\varepsilon^i = \frac{\delta S}{\delta\Phi_i} = 0,$$

but are different off-shell. The situation is qualitatively the same for the gauge-fixing dependence.

B. Giacchini, T. de Paula Netto, I.Sh., PRD 102 (2020), 2006.04217.

For the effective QG based on Einstein's GR with the cosmological constant, this prescription gives

$$\bar{\Gamma}^{(1)}_{\rm div} \,=\, -\frac{1}{\epsilon}\int d^4x\sqrt{-g}\,\Big\{\frac{121}{60}C^2 - \frac{151}{180}E + \frac{31}{36}R^2 + 8\Lambda R + 12\Lambda^2\Big\}\,.$$

This, completely invariant, result enables us to construct the renormalization group equations

$$\mu\frac{d}{d\mu}\Big(\frac{1}{16\pi G}\Big) \,=\, \frac{8\Lambda}{(4\pi)^2}\,, \qquad \mu\frac{d\Lambda}{d\mu} \,=\, -\frac{2(16\pi G)\Lambda^2}{(4\pi)^2}\,. \qquad (2)$$

The solutions can be easily found in the form ($\gamma_0 = 16\pi G_0\Lambda_0^2$).

$$G(\mu) \,=\, \frac{G_0}{\Big[1 + \frac{10}{(4\pi)^2}\,\gamma_0\,\ln\frac{\mu}{\mu_0}\Big]^{4/5}}\,, \qquad \Lambda(\mu) \,=\, \frac{\Lambda_0}{\Big[1 + \frac{10}{(4\pi)^2}\,\gamma_0\,\ln\frac{\mu}{\mu_0}\Big]^{1/5}}.$$

We get well-defined runnings of the Newton and cosmological constants between the Planck and Hubble scales!

It is due to the QG in its effective version, because below the Planck scale all extra degrees of freedom get inactive.

The renormalization group (RG) equations in effective QG:

$$\mu \frac{d}{d\mu}\left(\frac{1}{16\pi G}\right) = \frac{8\Lambda}{(4\pi)^2}\,, \qquad \mu \frac{d\Lambda}{d\mu} = -\frac{2(16\pi G)\Lambda^2}{(4\pi)^2}\,. \tag{2}$$

The two most remarkable properties of these equations and their solutions are as follows:

i) Universality, i.e., Eqs. (2) do not depend on the gauge fixing, parametrization of quantum fields or any kind of hypothesis and assumptions, except the definition of the VdW effective action.

ii) Can be regarded exact, i.e., not restricted to one-loop order. All higher-loop corrections are suppressed by the powers of

$$\frac{\Lambda}{M_P^2} = \frac{\rho_\Lambda}{M_P^4}.$$

In the present-day Universe this quantity is of the order of 10^{-120}, **but even in the inflationary epoch it is at least** 10^{-12}.

Thus, RG equations (2) **describe an effectively exact running.**

The full action including active, lowest derivative and inactive higher derivative terms has the form

$$S_{\rm tot} \,=\, \int d^4x\sqrt{-g}\,\Big\{ -\frac{1}{\kappa^2}\big(R+2\Lambda\big) - \frac{1}{2\lambda}C^2 + \frac{1}{2\rho}E - \frac{1}{2\xi}R^2$$
$$+\frac{1}{2\zeta}\,C_{\mu\nu\alpha\beta}\,C^{\alpha\beta}_{\cdot\;\cdot\;\rho\sigma}\,C^{\rho\sigma\mu\nu} + \sum_{n=1}^{N}\Big[\omega_{n,C}C\Box^n C + \omega_{n,R}R\,\Box^n R\Big] + \,...\Big\},$$

onde λ, ρ, ξ are the dimensionless parameters of the action.

The renormalization of the higher derivative terms in the effective QG performs similar to the renormalization of the vacuum action of gravity in the semiclassical approach.

At the one-loop level we get exact beta functions for the fourth-derivative terms, e.g.,

$$\mu\frac{d\lambda}{d\mu} \,=\, -\frac{a^2}{(4\pi)^2}\,\lambda^2\,, \qquad a^2 \,=\, a^2_{\rm QG} + \frac{1}{5} + \frac{N_f}{10}, \qquad a^2_{\rm QG} = \frac{121}{30}.$$

where N_f is the number of fermions.

Other exact equations coming from the one-loop calculations:

$$\mu\frac{d\rho}{d\mu} = -\frac{c^2}{(4\pi)^2}\,\rho^2, \quad c^2 = c^2_{\text{QG}} + \frac{31}{90} + \frac{11N_f}{180}, \quad c^2_{\text{QG}} = \frac{151}{90}$$

and

$$\mu\frac{d\xi}{d\mu} = -\frac{b^2}{(4\pi)^2}\,\xi^2, \qquad b^2 = b^2_{\text{QG}} = \frac{31}{18}.$$

It is easy to see that the last equation does not solve a physical problem of deriving the coefficient $5 \cdot 10^8$ of the R^2 term to fit what is required for the inflationary model of Starobinsky.

And this means that this problem cannot be resolved by the quantum effects of gravity.

Some other physical input is needed. And it is certainly useful to know this!

An example of the exact effective equation coming from the two-loop calculation is the one for the C^2term:

$$\bar{\Gamma}^{(2)}_{\rm div} \,=\, \frac{\mu^{n-4}}{(4\pi)^4(n-4)}\,\frac{209}{1440}\,\kappa^2 \int d^n x\sqrt{-g}\; C_{\mu\nu\alpha\beta}\,C^{\alpha\beta}_{\cdot\;\cdot\;\rho\sigma}\,C^{\rho\sigma\mu\nu}\,.$$

The beta function for the parameter ζ requires the two-loop calculation

$$\beta_\zeta \,=\, -\,\frac{a^2_{W3}}{(4\pi)^4}\,(16\pi G)\,\zeta^2, \qquad a^2_{W3} \,=\, \frac{209}{720}.$$

M.H. Goroff and A. Sagnotti, Nucl. Phys. B266 (1986).

The contributions of three- and higher-loop diagrams are suppressed by the powers of $G\Lambda \sim \frac{\rho_\Lambda}{M_P^4}$ which is numerically a very small quantity, at least below the Planck energy scale.

Thus, all these equations can be safely regarded exact.

Conclusions

- **The construction of QG theory which is not restricted to the IR region, is not possible without higher derivative terms.**

- **Including more than four derivatives provides theoretical advantages: superrenormalizable QG and well-defined exact renormalization group (RG) flow, free from usual ambiguities.**

- **Below the Planck scale, the massive modes of gravity decouple and the superrenormalizable QG - based running is not applicable.**

- **However, in the region below the mass spectrum of the fundamental higher derivative gravity, we meet an effective QG, which is remarkable in several respects.**

- **Assuming the Vilkovisky-DeWitt unique effective action, we arrive at the well-defined RG equations, which turn out exact, in the sense they are free from the higher-loop corrections.**